

Reversible nilpotent centers with cubic homogeneous nonlinearities

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Abstract

We provide 13 non-topological equivalent classes of global phase portraits in the Poincaré disk of reversible cubic homogeneous systems with a nilpotent center at origin, which complete the classification of the phase portraits of the nilpotent centers with cubic homogeneous nonlinearities.

Keywords: two dimensional differential systems, nilpotent centers, cubic polynomial differential systems, phase portrait

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1. Introduction and statement of the results

One of the main problems in the qualitative theory of planar polynomial differential systems, beside determining their limit cycles and their number, is the center-focus problem, i.e. the problem of distinguishing between a center or a focus. The beginnings of this problem goes back to Poincaré, who defined a *center* as a singular point with a neighbourhood filled with periodic orbits except the singular point.

It is known that if the polynomial differential system has a center at the origin, then there exists a change of variables and a time rescaling (if

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necessary) which transforms the original system in one of the followings

$$\dot{x} = -y + P(x, y), \quad \dot{y} = x + Q(x, y); \quad (1)$$

$$\dot{x} = y + P(x, y), \quad \dot{y} = Q(x, y); \quad (2)$$

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y); \quad (3)$$

where $P(x, y)$ and $Q(x, y)$ are polynomials without constant and linear terms. The center of the form (1) is called *linear type center*, of the form (2) *nilpotent center*, and of the form (3) *degenerate center*.

The complete classification of centers of the form (1) for quadratic real polynomial differential systems has been done mostly by Dulac [8], Kapteyn [16, 17] and Bautin [2]. The phase portraits of these systems were done by Vulpe [21] and Schlomiuk [20]. We know some classifications of centers for some families of cubic differential systems and of differential systems of higher degree, see [14, 19] and the references therein. The normal forms and the global phase portraits in the Poincaré disk for all the Hamiltonian linear type centers of linear plus cubic homogeneous planar polynomial vector fields have been given in [6].

In this paper we focus our attention on nilpotent centers. An algorithm that characterizes nilpotent centers and some others classes of degenerate centers has been given in [10, 11], see also [15]. It is known that quadratic polynomial differential systems has no nilpotent centers, see for instance [4]. There are works where the analytic integrability of nilpotent singular points has been studied, see [3, 4, 5, 12, 13].

The objective of this paper is to classify the global phase portraits of the nilpotent centers of the cubic polynomial differential systems of the form

$$\dot{x} = y + Ax^2y + Bxy^2 + Cy^3, \quad \dot{y} = -x + Px^2y + Kxy^2 + Ly^3. \quad (4)$$

Andreev et al. in [1] have obtained the normal forms for system (4) having a nilpotent center, see Theorem 1 of [1]. In fact there are two families of nilpotent centers, the Hamiltonian one, studied in [7], and the reversible family

$$\begin{aligned} \dot{x} &= y + Ax^2y + Cy^3, \\ \dot{y} &= -x^3 + Kxy^2. \end{aligned} \quad (5)$$

Note that this system is invariant, up to a time-reversal, under the sym-

metries with respect to both axes, and consequently also by the symmetry with respect to the origin. More precisely, the changes of variables $(x, y, t) \rightarrow (-x, y, -t)$, $(x, y, t) \rightarrow (x, -y, -t)$ and $(x, y) \rightarrow (-x, -y)$ leave invariant the system 5. So knowing the phase portrait of the system in one quadrant of the plane, we know completely the phase portrait of the system.

A first integral of system (5) has the form

$$H(x, y) = l_1^{-A-K+\sqrt{D}} l_2^{A+K+\sqrt{D}}, \quad (6)$$

where

$$l_1 = K(A + \sqrt{D} - K)(1 + Ax^2) + 2C^2y^2 + C(2 + (\sqrt{D} - K)x^2 + A(x^2 + 2Ky^2))$$

and

$$l_2 = K(-A + \sqrt{D} + K)(1 + Ax^2) - 2C^2y^2 + C(-2 + (\sqrt{D} + K)x^2 - A(x^2 + 2Ky^2))$$

are invariant curves of system (5) and $D = (A - K)^2 - 4C$.

For cubic systems with homogeneous nonlinearities with a nilpotent center at the origin until now only the global phase portraits of Hamiltonian centers were done, see [7]. Hence our goal for completing the classification of the nilpotent centers of the cubic polynomial differential systems (4) is to obtain the global phase portraits in the Poincaré disk of systems (5). To do this we will use the Poincaré compactification of polynomial vector fields, see section 2. Two vector fields on the Poincaré disk are *topologically equivalent* if there exists a homeomorphism from one onto the other which sends orbits to orbits preserving or reversing the direction of the flow. We shall provide in Figure 1 the distinct topologically equivalent phase portraits of system (5), giving only their separatrix skeleton, see a precise definition at the end of section 2.

Our main result is following

Theorem 1. *The global phase portraits of system (5) with a center at the origin are topologically equivalent to one of the phase portraits given in Figure 1.*

The proof of Theorem 1 is analytical, and provides the existence of the 13 global phase portraits given in the statement of Theorem 1 and no others,

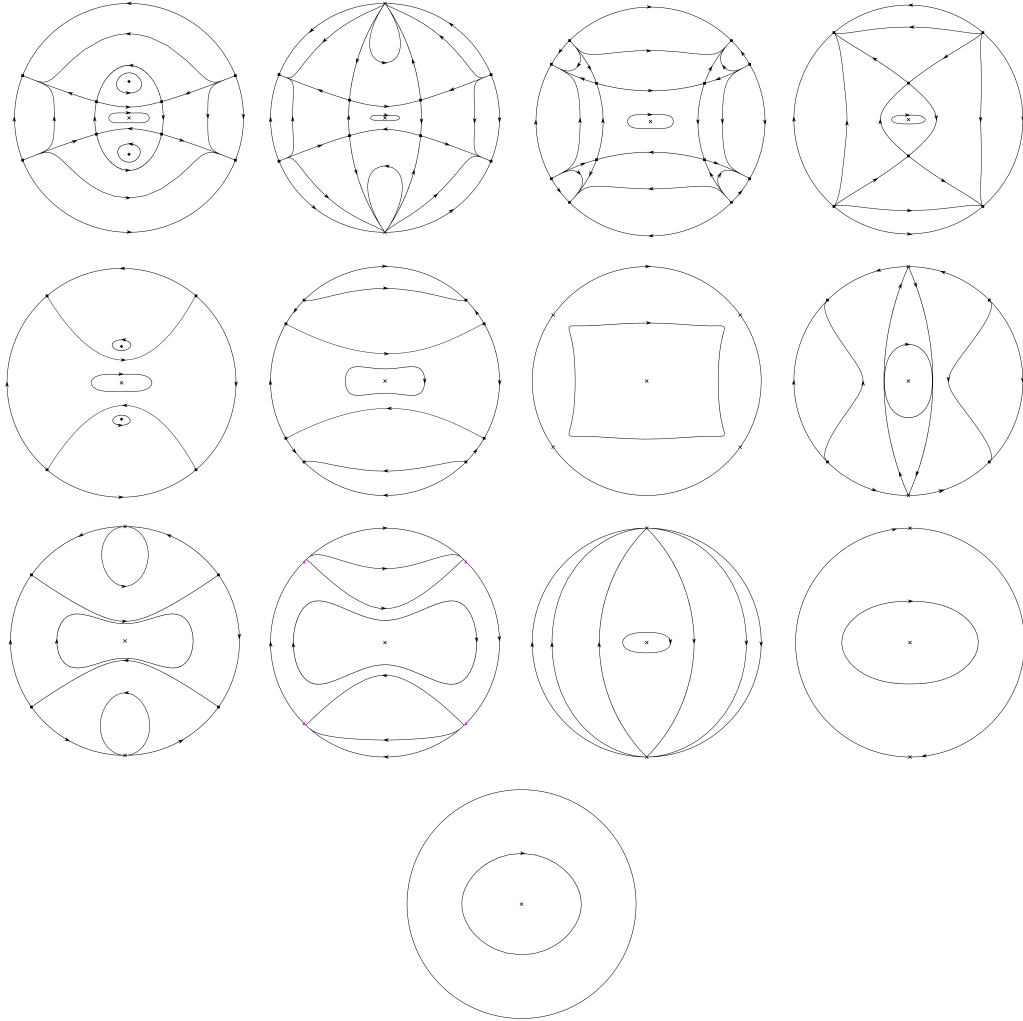


Figure 1: Global phase portraits of system (5).

depending on the parameters A , C and K of system (5). Classification of global phase portraits depending on parameters is presented in the next.

Cases and conditions on parameters A, K, C and D

I.	$C < 0$	$AK + C < 0$	$K > 0$				
II.	$C = 0$	$AK + C < 0$	$K > 0$				
III.	$C > 0$	$AK + C < 0$	$K > 0$		$D > 0$		
IV.	$C < 0$		$K \leq 0$				
V.	$C < 0$	$AK + C \geq 0$	$K > 0$				
VI.	$C > 0$	$AK + C \geq 0$			$D > 0$	$-A + K - \sqrt{D} > 0$	
VII.	$C > 0$	$AK + C = 0$	$K > 0$				$A = -K$
VIII.	$C = 0$		$K \leq 0$				$A < K$
IX.	$C = 0$			$A \geq 0$			$A < K$
X.	$C > 0$	$AK + C \geq 0$		$A \geq 0$	$D = 0$		$A < K$
XI.	$C = 0$			$A < 0$			$A \geq K$
XII.	$C = 0$			$A \geq 0$			$A \geq K$
XIII.	$C > 0$	$AK + C \geq 0$			$D \geq 0$		$A > K$
					$D < 0$		
					$D > 0$	$-A + K + \sqrt{D} < 0$	
		$AK + C < 0$	$K < 0$		$D > 0$		$A > K$

We must mention that the existence of these 13 global phase portraits has also been checked using the program P4, which do the global phase portrait of a given polynomial differential system in the Poincaré disk. For more details on the program P4, see Chapters 9 and 10 of [9].

The rest of this paper is divided into four sections. In section 2 we have summarized the Poincaré compactification, in the next two sections we study the finite and infinite singular points of system (5). Finally in section 5 we prove Theorem 1.

2. The Poincaré compactification

Let

$$\mathcal{X} = (P(x, y), Q(x, y)) \tag{7}$$

be a polynomial vector field of degree d , i.e. d is the maximal degree of the polynomials P and Q . For studying the behaviour of the trajectories of a polynomial vector field (7) near infinity we use the Poincaré compactification. We denote by \mathbb{S}^2 the *Poincaré sphere*, which is the set of points $(s_1, s_2, s_3) \in \mathbb{R}^3$ such that $s_1^2 + s_2^2 + s_3^2 = 1$. Each polynomial vector field (7) can be extended analytically to the Poincaré sphere. By projection of each point

$x \in \mathbb{R}^2 \equiv \{(x_1, x_2, 1) \in \mathbb{R}^3\}$ onto the Poincaré sphere using the straight line through x and the origin of \mathbb{R}^3 we obtain two copies of the vector field \mathcal{X} on \mathbb{S}^2 , one in the north hemisphere and the other in the south. The equator $\mathbb{S}^1 = \{(s_1, s_2, s_3) \in \mathbb{S}^2; s_3 = 0\}$ corresponds to the infinity of \mathbb{R}^2 .

In order to describe the global phase portrait of a vector field \mathcal{X} it is necessary to study the finite singular points in $\mathbb{S}^2 \setminus \mathbb{S}^1$ and the infinite singular points in \mathbb{S}^1 .

For studying the compactified vector field $P(\mathcal{X})$ of \mathcal{X} on the Poincaré sphere we use six local charts; $U_i = \{s \in \mathbb{S}^2; s_i > 0\}$ and $V_i = \{s \in \mathbb{S}^2; s_i < 0\}$ for $i = 1, 2, 3$. The expressions for the corresponding vector fields on \mathbb{S}^2 in the local charts U_i are

$$\text{in } U_1 : \dot{u} = v^d \left[-uP\left(\frac{1}{v}, \frac{u}{v}\right) + Q\left(\frac{1}{v}, \frac{u}{v}\right) \right], \dot{v} = -v^{d+1}P\left(\frac{1}{v}, \frac{u}{v}\right); \quad (8)$$

$$\text{in } U_2 : \dot{u} = v^d \left[P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right) \right], \dot{v} = -v^{d+1}Q\left(\frac{u}{v}, \frac{1}{v}\right); \quad (9)$$

$$\text{in } U_3 : \dot{u} = P(u, v), \dot{v} = Q(u, v). \quad (10)$$

The expression for V_i is the same than for U_i multiplied by $(-1)^{d-1}$ for $i = 1, 2, 3$. So we only need to study the phase portrait on U_i for $i = 1, 2, 3$. Finite singular points can be studied using U_3 , while for the infinite singular points it is sufficient to look at $U_1|_{v=0}$ and at the origin of U_2 . The projection of $(s_1, s_2, s_3) \rightarrow (s_1, s_2)$ of the northern hemisphere of the Poincaré sphere on the equator plane define the *Poincaré disk* \mathbb{D}^2 . On this disk we present the phase portraits of our system (5).

It is well known that the separatrices of a compactified polynomial vector field $P(\mathcal{X})$ in the Poincaré disk are the singular points, the limit cycles, the separatrices of its hyperbolic sectors, and all the orbits at infinity (i.e. \mathbb{S}^1), for more details see [9].

Assume that $P(\mathcal{X})$ has finitely many separatrices, and let \sum be the set of all separatrices of $P(\mathcal{X})$ on the Poincaré disk \mathbb{D}^2 . Then the open components of $\mathbb{D} \setminus \sum$ are called the *canonical regions* of $P(\mathcal{X})$. The *separatrix skeleton* of $P(\mathcal{X})$ is the set formed by all the separatrices (i.e. by \sum) and one orbit for each canonical region.

The next theorem shows that it is sufficient to draw a separatrix skeleton of $P(\mathcal{X})$ in the Poincaré disk \mathbb{D}^2 for determining completely the phase portrait of $P(\mathcal{X})$, see [18].

Theorem 2. (*Neumann Theorem*) Assume that $P(\mathcal{X}_1)$ and $P(\mathcal{X}_2)$ are two compactified polynomial vector fields having finitely many singular points in the Poincaré disk \mathbb{D}^2 . Then their phase portraits are topologically equivalent if and only if their separatrix skeletons are homeomorphic.

For more details on the Poincaré compactification see Chapter 5 of [9].

3. The finite singular points

We classify a singular point p (finite or infinite) as follows:

- p is *hyperbolic* if its two eigenvalues have non-zero real part. The local phase portraits of hyperbolic singular points can be obtained from Theorem 2.15 of [9].
- p is *semi-hyperbolic* if one of its eigenvalues is zero and the other is non-zero. The local phase portraits of semi-hyperbolic singular points are characterized in Theorem 2.19 of [9].
- p is *nilpotent* if its two eigenvalues are zero but its linear part is not identically zero. The local phase portraits of nilpotent singular points can be obtained from Theorem 3.15 of [9].
- p is *degenerate* if its linear part is identically zero. The local phase portraits of such singular points are studied doing changes of variables called blow-ups, see Chapters 2 and 3 of [9].

System (5) has a center at the origin, as it was proved in [1]. Parameters A, K and C of this system determine the number of finite singular points of the system and their local phase portraits. Depending on the parameters we can have 1, 3, 5 or 7 finite singular points. To compute the finite singular points it is necessary to solve the system

$$y(1 + Ax^2 + Cy^2) = 0, \quad x(-x^2 + Ky^2) = 0.$$

Case 1: $C < 0$, $AK + C < 0$ and $K > 0$. Then system (5) has 7 finite singular points

$$p_1 = (0, 0), \quad p_{2,3,4,5} = \left(\pm \sqrt{-\frac{K}{AK + C}}, \pm \sqrt{-\frac{1}{AK + C}} \right), \quad p_{6,7} = \left(0, \pm \sqrt{-\frac{1}{C}} \right).$$

There is a center at the origin, $p_{2,3,4,5}$ are hyperbolic saddles, and $p_{6,7}$ are either centers or foci. The first integral (6) in the singular points $p_{6,7}$ is well defined under these conditions, hence the points are always centers in this case.

Case 2: $C \geq 0$, $AK + C < 0$ and $K > 0$. Then there are 5 finite singular points

$$p_1 = (0, 0), \quad p_{2,3,4,5} = \left(\pm \sqrt{-\frac{K}{AK + C}}, \pm \sqrt{-\frac{1}{AK + C}} \right).$$

The finite singular point p_1 is center and the others are hyperbolic saddles.

Case 3: $C < 0$ and $AK + C \geq 0$; or $C < 0$, $AK + C < 0$ and $K \leq 0$. Then there are only 3 finite singular points

$$p_1 = (0, 0) \text{ and } p_{2,3} = \left(0, \pm \sqrt{-\frac{1}{C}} \right).$$

There is a center at the origin, but the nature of singular points $p_{2,3}$ depends on parameters. If $C < 0$, $AK + C \geq 0$ and $K > 0$, then $p_{2,3}$ are either centers or foci. Since the first integral, as in case 1, is well defined, singular points $p_{2,3}$ are always centers. In case $C < 0$ and $K = 0$ the points $p_{2,3}$ are degenerated points. For the other conditions the finite singular points $p_{2,3}$ are hyperbolic saddles.

Case 4: $C \geq 0$ and $AK + C \geq 0$ or $C \geq 0$, $AK + C < 0$ and $K < 0$. Then the origin is the only finite singular point, and it is a center.

4. The infinite singular points

We study first the infinite singular points on the local chart U_1 (and V_1). From (8) system (5) in the local chart U_1 becomes

$$\begin{aligned} \dot{u} &= -1 - u^2(A - K + Cu^2 + v^2), \\ \dot{v} &= -uv(A + Cu^2 + v^2). \end{aligned} \tag{11}$$

For each real root u^* of the polynomial $f(u) = \dot{u}|_{v=0} = -1 - (A - K)u^2 - Cu^4$, the point $(u^*, 0)$ is a singular point in U_1 . Since

$$f(u) = -C(u + r_1)(u - r_1)(u + r_2)(u - r_2),$$

with

$$r_1 = \sqrt{\frac{K - A + \sqrt{D}}{2C}}, \quad r_2 = \sqrt{\frac{K - A - \sqrt{D}}{2C}} \quad \text{and} \quad D = (A - K)^2 - 4C,$$

it follows that in the infinity of the local chart U_1 there can be 0, 2 or 4 singularities.

The eigenvalues λ_1 and λ_2 for these infinite singular points (if they are real) are

- for $(u, v) = (r_1, 0)$ we have $\lambda_1 = -r_1(A + Cr_1^2)$ and $\lambda_2 = -2\sqrt{D}r_1$;
- for $(u, v) = (-r_1, 0)$ we have $\lambda_1 = r_1(A + Cr_1^2)$ and $\lambda_2 = 2\sqrt{D}r_1$;
- for $(u, v) = (r_2, 0)$ we have $\lambda_1 = r_2(K + Cr_2^2)$ and $\lambda_2 = 2\sqrt{D}r_2$;
- for $(u, v) = (-r_2, 0)$ we have $\lambda_1 = -r_2(K + Cr_2^2)$ and $\lambda_2 = -2\sqrt{D}r_2$.

There are no infinite points at U_1 in the following cases:

- $D < 0$;
- $D = 0$ and $A > K$;
- $D > 0$, $C > 0$ and $K - A + \sqrt{D} < 0$;
- $D > 0$, $C < 0$ and $K - A - \sqrt{D} > 0$;
- $C = 0$ and $A \geq K$.

The system has two singular points in the local chart U_1 in the following cases:

- If $C < 0$ and $D > 0$ we have two hyperbolic nodes when $AK + C < 0$ and $K \geq 0$, and two hyperbolic saddles when $AK + C > 0$ and $K > 0$.
- If $D = 0$ and $K > A$ we have two degenerate singularities formed by two hyperbolic sectors (see section 5).
- If $C = 0$ and $A < K$ we have two hyperbolic nodes when $AK < 0$, and two hyperbolic saddles when $AK > 0$.

- If $C = A = 0$ and $K > 0$ we have two semi-hyperbolic saddles.
- If $C = K = 0$ and $A < 0$ we have two semi-hyperbolic nodes.

The system has four singular points in the local chart U_1 when $D > 0$, $C > 0$ and $-A + K - \sqrt{D} > 0$. The local phase portraits at these singular points are:

- two hyperbolic saddles and two hyperbolic nodes if $AK + C > 0$;
- four semi-hyperbolic singular points when $AK + C = 0$; and
- four hyperbolic nodes otherwise.

System (5) in the local chart U_2 becomes

$$\begin{aligned}\dot{u} &= C + u^2(A - K + u^2) + v^2, \\ \dot{v} &= uv(-K + u^2).\end{aligned}\tag{12}$$

If $C = 0$ then there is an infinite singular point at the origin of U_2 , else there is no singular point at the origin of U_2 . Since this infinite singular point has linear part identically zero the local phase portrait is obtained doing blow-ups. The local phase portraits that can appear are shown on Figure 2.

As it is seen in Figure 2 the origin of U_2 can consists of either two elliptic sectors and two parabolic sectors, or two elliptic sectors, or six hyperbolic sectors, or two hyperbolic sectors and two parabolic sectors, or just two hyperbolic sectors. For more details see section 5.

5. Global phase portraits

Taking into account together the informations about the finite and infinite singular points provided in sections 3 and 4, and the first integral (6) we shall find for system (5) 13 different phase portraits depending on the values of the parameters A , K and C .

5.1. Global phase portraits with seven finite singular points

(Case 1 of section 3). From section 4 we know that there are two infinite singular points in U_1 and since $C \neq 0$ the origin of U_2 is not an infinite singular point. The infinite singular points at the $U_1 \cup V_1$ are repelling/attracting

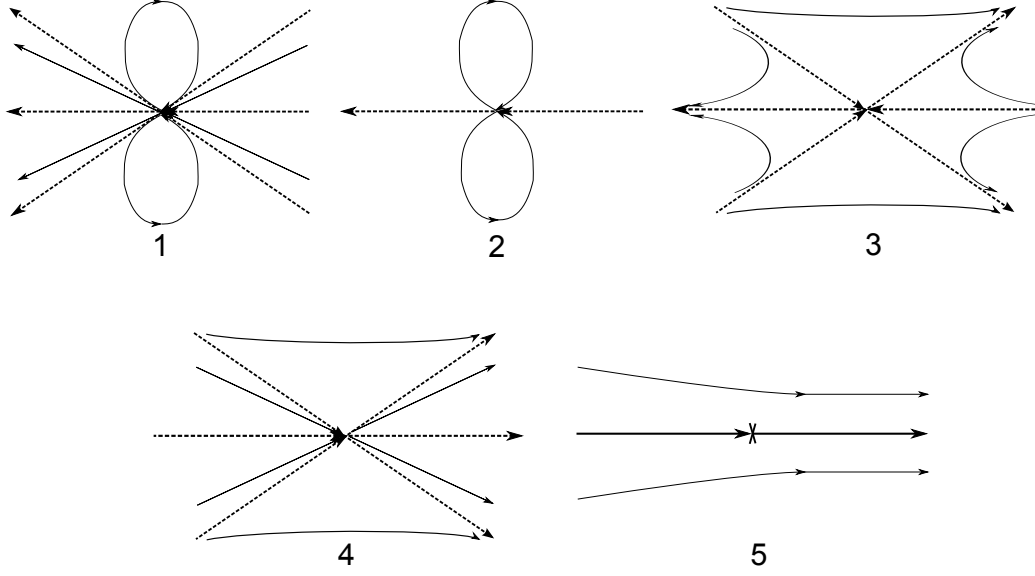


Figure 2: Local phase portraits at the origin of U_2

nodes. The finite singular points are a center at the origin, four saddles at the points

$$\left(\pm \sqrt{-\frac{K}{AK+C}}, \pm \sqrt{-\frac{1}{AK+C}} \right)$$

and two centers at the points

$$\left(0, \pm \sqrt{\frac{-1}{C}} \right).$$

Since the system is symmetric four finite points which are saddles are on the boundary of the period annulus of the center at the origin. The separatrices of these four saddles are forming in pairs a boundary of the period annuli of the centers at $(0, \sqrt{\frac{-1}{C}})$ and $(0, -\sqrt{\frac{-1}{C}})$. The unstable separatrix is connecting one saddle of this pair with an attracting node at the infinity, and the stable separatrix of the second saddle of this pair comes from a repelling node at the infinity. The connexion of the separatrices is determined using the first integral (6). The important fact about this global phase portrait is about limit cycles. The domains of definition of the first integral (6) and of the

first integral given by its inverse, together are \mathbb{R}^2 . Hence this prevents the existence of limit cycles, because if a limit cycle exists together with a first integral, the limit cycle lives where the first integral is not defined.

The global phase portraits of these systems are topologically equivalent to the phase portraits of Figure 3.

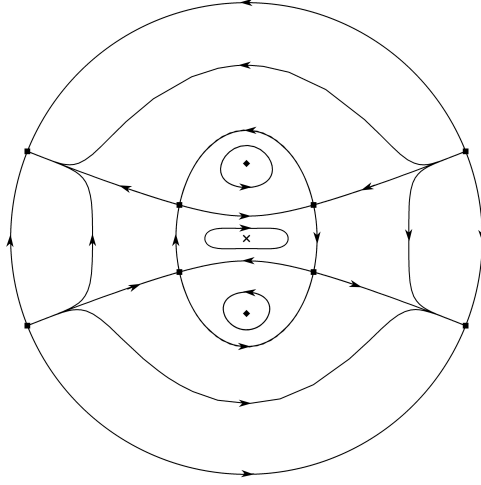


Figure 3: Phase portrait for $C = -2$, $A = -2$ and $K = 4$.

5.2. Global phase portraits with five finite singular points

(Case 2 of section 3). In this case system (5) has two different global phase portraits with five finite singular points.

The five finite singular points are the center at the origin and four saddles at the points $\left(\pm \sqrt{-\frac{K}{AK+C}}, \pm \sqrt{-\frac{1}{AK+C}} \right)$.

When $C = 0$, $A < 0$ and $K > 0$ there are two nodes at the infinity of U_1 , one is repelling and the other attracting, and there is one infinite singular point at the origin of U_2 , whose linear part is identically zero. Doing a blow-up in direction u , i.e. $(u, v) \mapsto (u, w)$ with $w = v/u$, we obtain the system

$$\begin{aligned} \dot{u} &= u^2(A - K + w^2 + u^2), \\ \dot{w} &= -uw(A + w^2). \end{aligned}$$

We eliminate the common factor u doing a rescaling of the independent variable. The new differential system on $u = 0$ has three singular points because $A < 0$: a saddle at the origin and two attracting nodes at $(0, \pm\sqrt{-A})$. The process of going back through the changes of variables provides the local phase portrait at the origin of U_2 given in the first phase portrait of Figure 2.

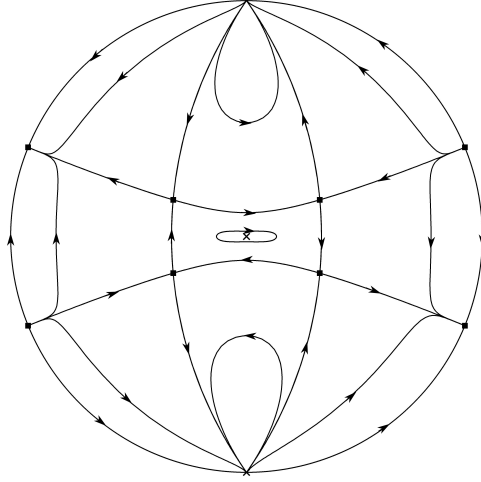


Figure 4: Phase portrait for $C = 0$, $A = -2$ and $K = 4$.

For the second case with five finite singular points, i.e. when the parameters satisfy $C > 0$, $AK + C < 0$, $K > 0$ and $D > 0$, at infinity there are four pairs of infinite singular points at $U_1 \cup V_1$. All four infinite singular points are nodes, two of them are repellers and two attractors. The global separatrices have been determined using the first integral. The corresponding global phase portrait is shown in Figure 5.

5.3. Global phase portraits with three finite singular points (Case 3 of section 3).

For the values of the parameters $C < 0$ and $K < 0$ the finite singular points are: a center at the origin and two saddles at $(0 \pm \sqrt{-\frac{1}{C}})$. The saddles are on the boundary of the period annulus of the center at the origin. In the local charts $U_1 \cup V_1$ there are two pairs of nodes. The global phase portrait of this system is topologically equivalent to the one of Figure 6.

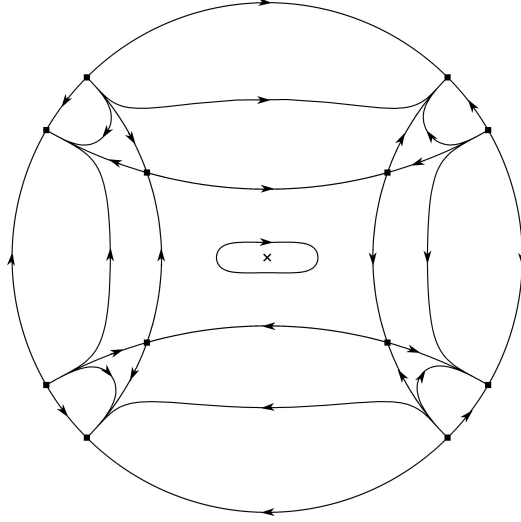


Figure 5: Phase portrait for $C = 3$, $A = -2$ and $K = 2$.

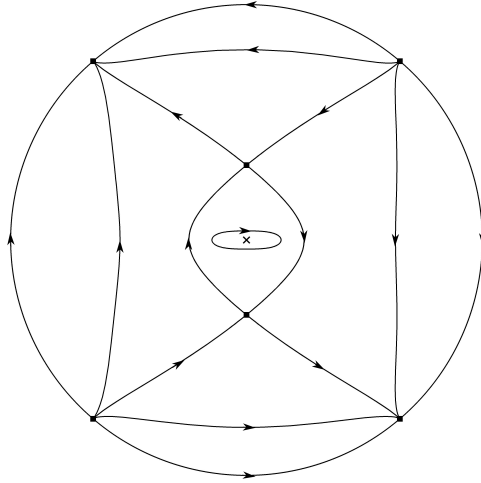


Figure 6: Phase portrait for $C = -2$, $A = -2$ and $K = -4$.

If parameters satisfy the conditions $C < 0$, $AK + C < 0$ and $K = 0$, then beside the center at the origin there are two degenerate finite singular points at $(0, \pm\sqrt{\frac{-1}{C}})$. Doing blow-ups we see that these finite singular points are

topological saddles. At $U_1 \cup V_1$ we have the only infinite singular points, two pairs of nodes. The global phase portrait is topologically equivalent to the previous one given in Figure 6.

When $C < 0$, $AK + C > 0$ and $K > 0$ there are three centers, at the origin and at $(0, \pm\sqrt{-\frac{1}{C}})$. The infinite singular points which are in $U_1 \cup V_1$ are saddles. The separatrices of the saddles at the infinity are forming period annulus of the centers. The corresponding global phase portrait is given in Figure 7.

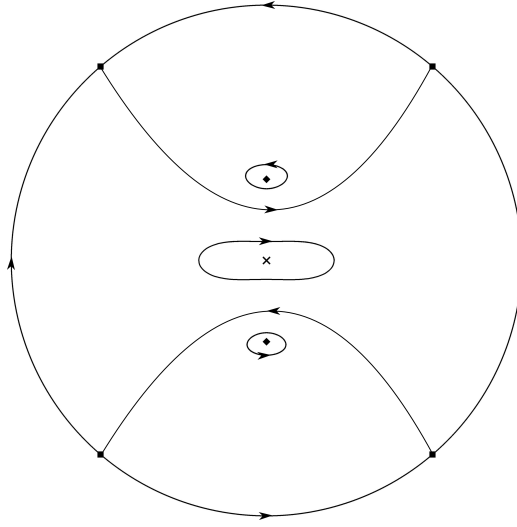


Figure 7: Phase portrait for $C = -2$, $A = 4$ and $K = 2$.

Finally for the values $C < 0$, $AK + C = 0$ and $K > 0$ there is a center at the origin and a center at $(0, \pm\sqrt{-\frac{1}{C}})$. At the $U_1 \cup V_1$ we obtain two semi-hyperbolic points, which are topological saddles. The corresponding global phase portrait is given in Figure 7.

5.4. Global phase portraits with one finite singular point

(Case 4 section 3). As we shall see there are several global phase portraits with only one finite singular point, which of course is a center at the origin.

For the values $D > 0$, $C > 0$, $AK + C \geq 0$, $-A + K + \sqrt{D} > 0$ and $-A + K - \sqrt{D} > 0$ we have at U_1 four singular points. Two of these infinite singular points are nodes and the other two are saddles. The origins of U_2 and V_2

are not singular points because $C \neq 0$. Depending on parameters, these four singular points at U_1 can be either a semi-hyperbolic saddle, a node, a node and a semi-hyperbolic saddle; or a node, a semi-hyperbolic saddle, a semi-hyperbolic saddle and a node. But there exists a homeomorphism, which transforms one phase portrait into the other, hence there two configurations providing global phase portraits which are topologically equivalent. The global phase portrait is as in Figure 8.

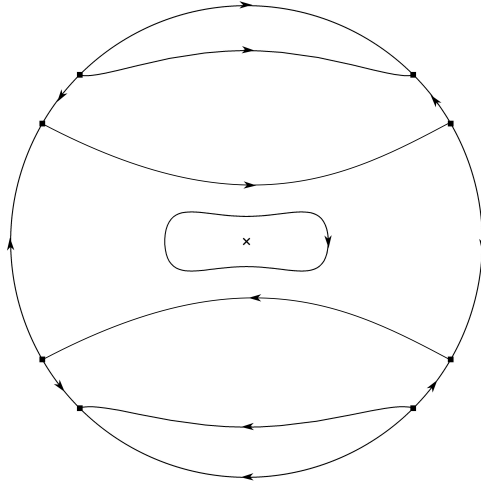


Figure 8: Phase portrait $C = 3$, $A = 0$ and $K = 4$

The system with parameters satisfying $C > 0$, $AK + C = 0$, $A + K = 0$ and $K > 0$ has two pairs of singular points $(\pm \sqrt{\frac{1}{K}}, 0)$ at $U_1 \cup V_1$. These two points need to be study with blow-ups because their linear parts are zero. First we translate the singular point $(\sqrt{\frac{1}{K}}, 0)$ at the origin of U_1 doing the change of variables $(u, v) \mapsto (z, v)$, after we do the blow-up $(z, v) \mapsto (z, w)$ where $w = v/z$ in direction u_1 . Now from the obtained differential system we eliminate the common factor z doing a rescaling of the independent variable. Thus we get the differential system

$$\begin{aligned}\dot{z} &= -4Kz - 4K^{\frac{3}{2}}z^2 - K^2z^3 - \frac{1}{K}zw^2 - 2\frac{1}{\sqrt{K}}z^2w^2 - z^3w^2, \\ \dot{w} &= 2Kw + K^{\frac{3}{2}}zw + \frac{1}{K}w^3 + \frac{1}{\sqrt{K}}zw^3.\end{aligned}$$

This system has a saddle at the origin. Going back through all the changes of variables, we get the local phase portrait at the singular point $(\sqrt{\frac{1}{K}}, 0)$, which is the one shown in Figure 9. The same study can be made at the singular point $(-\sqrt{\frac{1}{K}}, 0)$. Hence the global phase portrait of the system is shown in Figure 10.

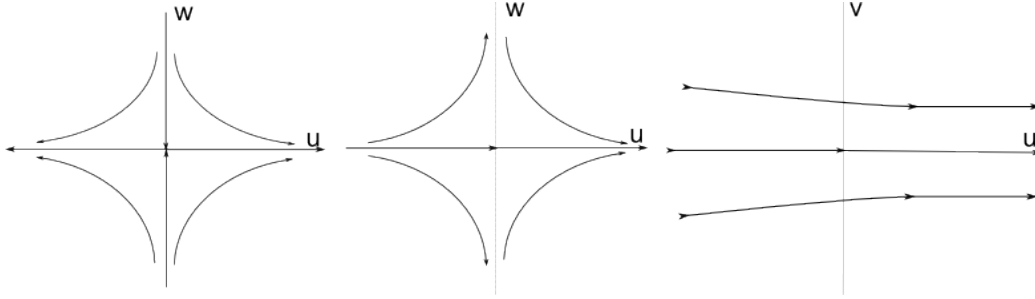


Figure 9: Blow-up of the origin of U_2 .

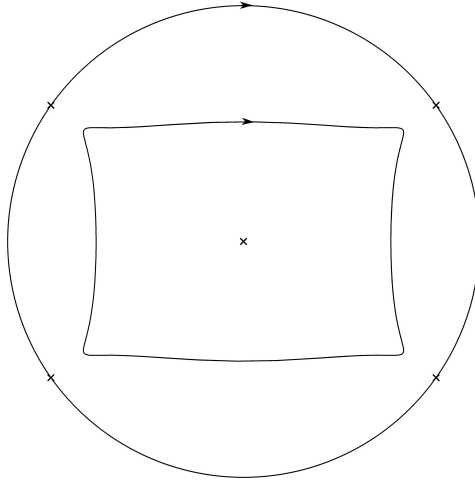


Figure 10: Phase portrait $C = 4$, $A = -2$ and $K = 2$.

For the values $C = K = 0$ and $A < 0$ there are two pairs of infinite singular points in $U_1 \cup V_1$, both are nodes. The origins of U_2 and V_2 are also

infinite singular points. To know the local phase portrait at the origin of U_2 we need to do a blow-up in the direction u , after to eliminate the common factor u the obtained differential system has three singular points on $u = 0$: a saddle at $(0, 0)$ and at $(0, \pm\sqrt{-A})$ two semi-hyperbolic saddles. Going back to the changes of variables we obtain the local phase portrait at the origin U_2 , see the second phase portrait of Figure 2. The global phase portrait of the system is topologically equivalent to the one of Figure 11.

The same phase portrait is obtained for the values $C = 0$, $K < 0$ and $A < K$.

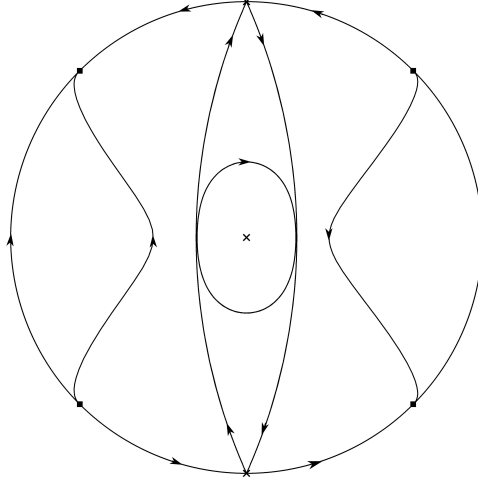


Figure 11: Phase portrait $C = 0$, $A = -5$ and $K = -4$.

For the values $C = 0$, $A > 0$, $K > 0$ and $K > A$ there are two singular points at U_1 , which are saddles. The origin of U_2 is also a singular point whose local phase portrait needs to be explored doing a blow-up in the u direction, obtaining the differential system:

$$\begin{aligned} \dot{u} &= u^2(A - K + w^2 + u^2), \\ \dot{w} &= -wu(A + w^2). \end{aligned} \tag{13}$$

By eliminating the common factor u the resulting differential system has one singular point on $u = 0$, the $(0, 0)$, which is an attracting node. Going back through the changes of variables we obtain the local phase portrait at

the origin of U_2 , see the third phase portrait of Figure 2. The global phase portrait of the system is topologically equivalent to the one of Figure 12.

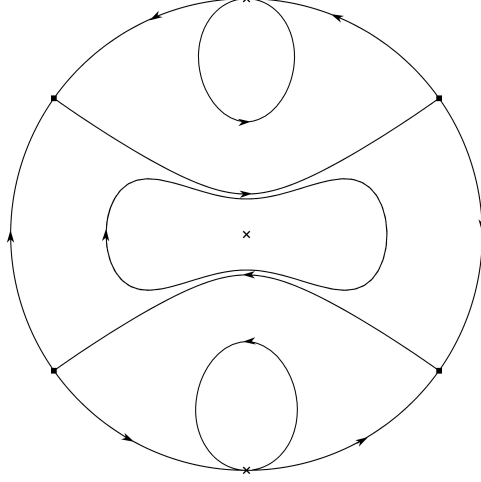


Figure 12: Phase portrait for $C = 0$, $A = 2$ and $K = 4$.

For the values of the parameters $C = 0$, $A = 0$ and $K > 0$ the system has as in the previous case two pairs of singular points at $U_1 \cup V_1$ which are semi-hyperbolic saddles. The origin of U_2 is also a singular point, doing a blow-up in direction u and working as in the previous case we obtain that the global phase portrait of this system is also given in Figure 12.

For the values of the parameters $D = 0$, $C > 0$, $K > A$ and $AK + C \geq 0$ the system has two pairs of infinite singular points at $U_1 \cup V_1$, and the origin of U_2 is not a singular point. All infinite singular points are semi-hyperbolic saddles-nodes. The global phase portrait of the system is topologically equivalent to the one of Figure 13.

For the values of the parameters $C = 0$, $A < 0$ and $K < A$ beside the center at the origin, there is a singular point at the origin of U_2 , its local phase portrait is studied doing blow-ups, see the fourth phase portrait of Figure 2). The global phase portrait of the system is given in Figure 14.

For the values of the parameters $C = 0$, $A = K$ and $A < 0$ there is only one finite singular point at the origin, and one pair at $U_2 \cup V_2$. The local phase portrait at the origin of U_2 is as in the previous case, hence we get the same global phase portrait, the one of Figure 14.

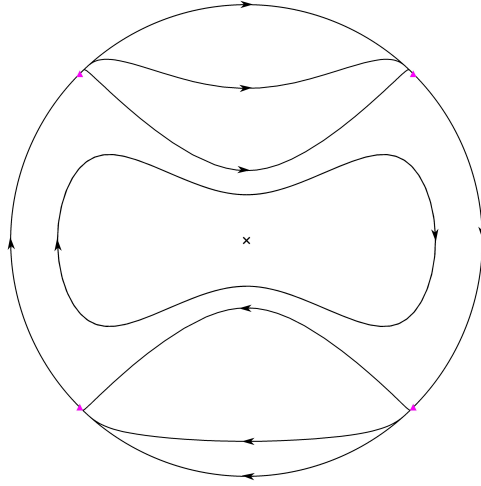


Figure 13: Phase portrait $C = 1$, $A = 2$ and $K = 4$

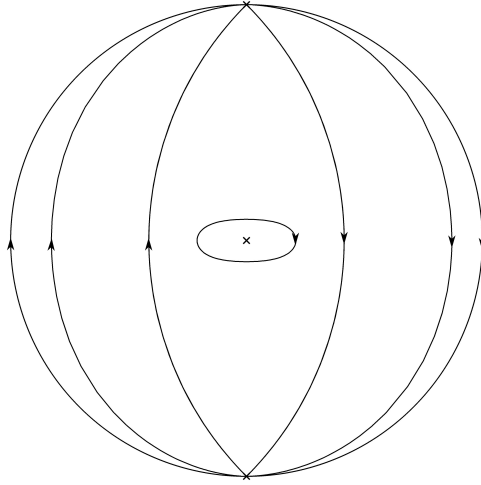


Figure 14: Phase portrait $C = 0$, $A = -1$ and $K = -3$.

For the values either $C = 0$, $A > 0$ and $0 \leq K \leq A$; or $C = 0$, $A > 0$ and $K < 0$; or $C = K = 0$ and $A \geq 0$; or $C = A = 0$ and $K < 0$, at $U_1 \cup V_1$ there are no singular points. The singular point at the origin of U_2 is degenerate, so

we must do blow-ups. If $C = A = 0$ and $K < 0$ after a blow-up in direction u and the elimination of a common factor u we obtain a semi-hyperbolic saddle at $u = 0$. Going back through the changes of variables, we obtain the fifth phase portrait of Figure 2. In the other cases we obtain, after the elimination of a common factor u , a saddle. Going back through the changes of variables, we obtain the phase portrait of Figure 9. In any case the global phase portrait always is topologically equivalent to the one of Figure 15.

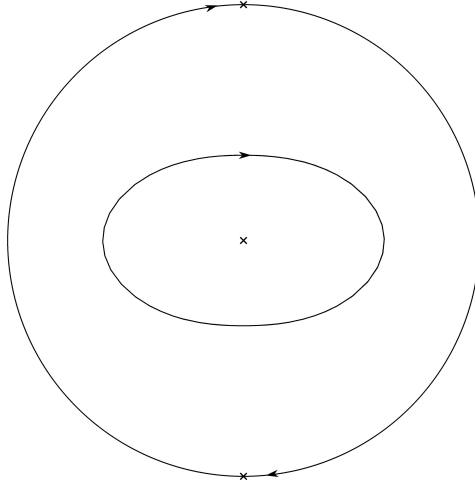


Figure 15: Phase portrait for $C = 0$, $A = 4$ and $K = 0$.

If the parameters of the system (5) are:

- either $D = 0$, $C > 0$, $K < A$ and $AK + C \geq 0$;
- or $D < 0$, $C > 0$ and $AK + C \geq 0$;
- or $D > 0$, $C > 0$, $A > K$ and $AK + C \geq 0$;
- or $D > 0$, $C > 0$, $A > K$, $K < 0$ and $AK + C < 0$;
- or $D > 0$, $C > 0$, $-A + K + \sqrt{D} < 0$ and $AK + C \geq 0$;

then there is only one finite singular point, the center at the origin, and there are no infinite singular points. The phase portrait is given in Figure 16.

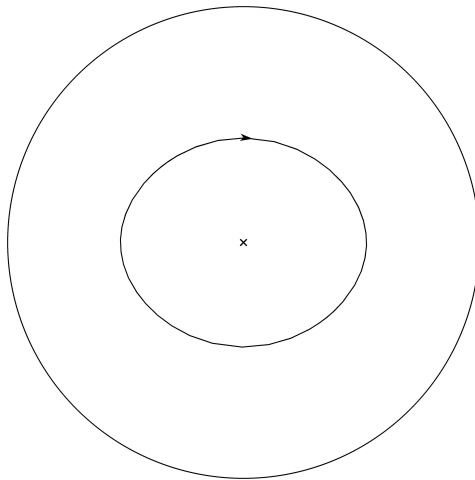


Figure 16: Phase portrait $C = 2$, $A = 1.5$ and $K = -2$.

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